

8 - Permutations and Combinations

To count permutations of indistinct letters, divide by the ways identical letters could be permuted:

Example 1. How many rearrangements of the word "MOMMA" are there?

$M_1OM_2M_3A$	$M_1OM_3M_2A$	$M_2OM_1M_3A$	$M_2OM_3M_1A$	$M_3OM_1M_2A$	$M_3OM_2M_1A$	→	MOMMA
$M_1M_2M_3AO$	$M_1M_3M_2AO$	$M_2M_1M_3AO$	$M_2M_3M_1AO$	$M_3M_1M_2AO$	$M_3M_2M_1AO$	→	MMMAO
$M_1M_2M_3OA$	$M_1M_3M_2OA$	$M_2M_1M_3OA$	$M_2M_3M_1OA$	$M_3M_1M_2OA$	$M_3M_2M_1OA$	→	MMMOA
⋮	⋮	⋮	⋮	⋮	⋮		

Answer: If M's are distinct, there are 5! such words. But the 3 M's are considered identical, so we group their 3! rearrangements together.

Answer: 5!/3!

Example 2. How many rearrangements of the word "STATISTICS" are there?

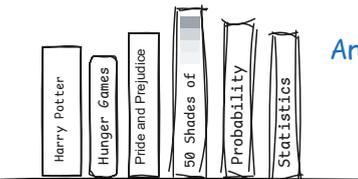
Answer: Of the 10 letter, there are 3 S's, 3 T's, 2 I's, so $\frac{10!}{3!3!2!}$

Example 3. How many rearrangements of 2 identical red books and 3 identical blue books on a bookshelf are there?

Answer: We count rearrangements of the word RRBBB or $\frac{5!}{3!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 2} = 10$

Count the unordered selection of k things from n things via "Yes"/"No" assignment:

Example 4. How many ways can you select 3 books from 6 books to bring with you?



Answer: Selecting 3 books from 6 is encoded as 3 Y's and 3 N's e.g. selecting first 3 books is encoded as "YYYN NN". Number of such selections = permutations of "YYYN NN" = 6!/(3!3!)

Definition. k-combination counts the unordered selection of k things from n distinct things, sampled without replacement:

$$C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \leftarrow \begin{array}{l} \text{"binomial coefficient"} \\ \text{or "n choose k"} \end{array}$$

Example 5. How many ways can you select 2 villagers from a district of 8,000 in the Hunger Games?

Answer: This is 8000 choose 2: C(8000,2).

Example 6. Flip 7 coins. How many outcomes have exactly 3 heads?

Answer: This is 7 choose 3, or # of permutations of the string "HHHTTTT".

Example 7. Permutation or combination?

- (a) Select and form a committee of 3 students from 110 students. $C(110,3)$
- (b) Hand out 1st, 2nd, and 3rd place awards to 3 of 40 contestants. $P(40,3)$
- (c) Selecting the three toppings out of 10 options for your pizza. $C(10,3)$
- (d) A team of 8 basketball players chooses a captain and co-captain. $P(8,2)$

Example 8. How many ways can a committee of 2 freshmen and 3 seniors be formed from 10 freshmen and 6 seniors?

Answer: (from 10 freshmen, choose 2) \times (from 6 seniors, choose 3)
 $= C(10,2) \times C(6,3)$

Example 9. How many 5-card hands have exactly ...

- (a) 1 ace and 2 jacks? (get 1 of 4 A's) \times (get 2 of 4 J's) \times (get 2 of 44 non-(A&J)'s)
 $= C(4,1) \times C(4,2) \times C(44,2)$.
- (b) 2 aces? (from 4 aces, choose 2) \times (from 52-4=48 non-aces, choose 3 cards)
 $= C(4,2) \times C(48,3) = 6 \times (48 \times 47 \times 46 / (3 \times 2))$.

Optional Homework due March 4th or 5th.

Show your work. Answer without work receives no credit.

1. How many ways can letters in the word 'NEEDLESSLY' be rearranged?
2. In a horse race, how many possible outcomes are there for first, second, and third place if there are 27 horses racing?
3. A bag has 12 differently colored marbles. How many different color combos can be selected by simultaneously drawing three marbles from the bag?
4. Three different prizes will be awarded to 35 graduating students. If no student can receive more than one prize, in how many different ways could the prizes be awarded?
5. How many 5-card hands have no red cards?
6. How many 5-card hands have only heart cards?
7. A school dance committee is to consist of 2 freshmen, 3 sophomores, 4 juniors, and 5 seniors. If 6 freshmen, 7 sophomores, 7 juniors, and 7 seniors are eligible to be on the committee, in how many ways can the committee be chosen?
8. How many 8-bit strings have exactly two 1's? ("1100000", "01000100")

Non-Homework Problems.

9. How many 8-bit strings have at least three 1's?
10. (a) How many arrangements of the letters in BALLYNALALLY are possible?
 (b) Of these arrangements, how many have all the L's together?
 (c) Of these arrangements, how many have all the letters in alphabetical order?
11. How many 5-card hands have exactly 2 hearts or 3 clubs?